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The largest singletons of set partitions☆

Yidong Sun, Xiaojuan Wu

Department of Mathematics, Dalian Maritime University, 116026 Dalian, PR China

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ABSTRACT

Recently, Deutsch and Elizalde have studied the largest and the smallest fixed points of permutations. Motivated by their work, we consider the analogous problems in set partitions. Let $A_{n,k}$ denote the number of partitions of $\{1, 2, \ldots, n+1\}$ with the largest singleton $\{k+1\}$ for $0 \le k \le n$. In this paper, several explicit formulas for $A_{n,k}$, involving a Dobinski-type analog, are obtained by algebraic and combinatorial methods. Furthermore, many combinatorial identities involving $A_{n,k}$ and Bell numbers are presented by operator methods, and congruence properties of $A_{n,k}$ are also investigated. It is shown that the sequences $(A_{n+k,k})_{n\ge 0}$ and $(A_{n+k,k})_{k\ge 0}$ (mod p) are periodic for any prime p, and contain a string of $p-1$ consecutive zeroes. Moreover their minimum periods are conjectured to be $N_p = \frac{p^p-1}{p-1}$.

1. Introduction

A *partition* of a set $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty and mutually disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. It is known that the number of partitions of $[n]$ with exactly k blocks is the Stirling number of the second kind $S(n, k)$, and the total number of partitions of $[n]$ is the nth Bell number B_n, which has the exponential generating function

$$B(x) = \sum_{n\ge 0} B_n \frac{x^n}{n!} = \exp(e^x - 1). \tag{1.1}$$

Differentiating (1.1) gives $B'(x) = e^x B(x)$, which leads to

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{1.2}$$

☆ Dedicated to L.C. Hsu, on the occasion of his ninetieth birthday.

E-mail address: sydmath@yahoo.com.cn (Y. Sun).

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A *singleton* of a partition is a block containing just one element. If $\{k\}$ is a singleton of a partition, we denote it by k for short. The number of partitions of $[n]$ without singletons is counted by V_n, where $(V_n)_{n\geq 0} = (1, 0, 1, 1, 4, 11, 41, 162, \ldots)$ [20], and has the exponential generating function

$$V(x) = \sum_{n\geq 0} V_n \frac{x^n}{n!} = \exp(e^x - x - 1). \tag{1.3}$$

Bernhart [3] has given a combinatorial interpretation for the relation $B_n = V_n + V_{n+1}$ which can also be obtained from $B(x) = V(x) + V'(x)$. By (1.1) and (1.3), one can deduce that

$$B_n = \sum_{j=0}^{n} \binom{n}{j} V_j \quad \text{and} \quad V_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} B_j.$$

Recently, Deutsch and Elizalde [6] have studied the largest and the smallest fixed points of permutations. Motivated by their work, we consider the analogous problems in set partitions. Let $A_{n,k}$ denote the number of partitions of $[n+1]$ with the largest singleton $k+1$. Clearly,

$$A_{n,0} = V_n \quad \text{and} \quad A_{n,n} = B_n.$$

This paper is organized as follows. In the next section, we find several explicit formulas for $A_{n,k}$, involving a Dobinski-type analog, by algebraic and combinatorial methods. In Section 3, we obtain many combinatorial identities involving $A_{n,k}$ and Bell numbers B_n by operator methods. In the last section, we consider the congruence properties of $A_{n,k}$ and Bell numbers B_n, find that the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ (modulo p) are periodic for any prime p and contain a string of $p-1$ consecutive zeroes. We also conjecture that their minimum periods are $N_p = \frac{p^p-1}{p-1}$ for any prime p.

2. The explicit formulas for $A_{n,k}$

It follows from the definition that

$$A_{n,k} = V_n + \sum_{j=0}^{k-1} A_{n-1,j}, \tag{2.1}$$

since by removing the largest singleton $k+1$ of a partition of $[n+1]$ containing singletons, we get a partition of $\{1, \ldots, k, k+2, \ldots, n+1\}$ whose largest singleton (if any) is less than $k+1$.

In (2.1), if we replace k by $k-1$, then by subtraction we obtain a recurrence for $n, k \geq 1$,

$$A_{n,k} = A_{n,k-1} + A_{n-1,k-1}. \tag{2.2}$$

Table 1 shows the values of $A_{n,k}$ for small n and k. It should be noticed that $\{A_{n+k,k}\}_{n\geq 0, k\geq 1}$ is just the Aitken array [20, A011971]. We point out that it is possible to give a direct combinatorial proof of the recurrence (2.2) from the definition of the $A_{n,k}$ for $n \geq k \geq 1$. Indeed, given a partition π of $[n+1]$ with the largest singleton $k+1$, if k is also a singleton, delete the singleton $k+1$ and subtract one from all the entries larger than $k+1$, to obtain a partition of $[n]$ with the largest singleton k; if k is not a singleton, exchange k and $k+1$, to obtain a partition of $[n+1]$ with the largest singleton k. When $k = 1$, (2.2) produces a new setting for Bell numbers, namely $A_{n+1,1} = A_{n,0} + A_{n+1,0} = V_n + V_{n+1} = B_n$. A simple combinatorial proof reads: given a partition π of $[n+2]$ with the largest singleton 2, if 1 is also a singleton, delete the two singletons 1, 2 and subtracting two from all the entries larger than 2, we obtain a partition of $[n]$ without singletons; if 1 is not a singleton, breaking the block containing 1 into singletons (more than one), then deleting the two singletons 1, 2 and subtracting two from all the entries larger than 2, we obtain a partition of $[n]$ with singletons.

The remainder of Section 2 will be devoted for developing various explicit formulas for $A_{n,k}$, as opposed to the recursive formula (2.2).

Table 1
The values of $A_{n,k}$ for n and k up to 7.

n/k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	1	1	2					
3	1	2	3	5				
4	4	5	7	10	15			
5	11	15	20	27	37	52		
6	41	52	67	87	114	151	203	
7	162	203	255	322	409	523	674	877

Theorem 2.1. *The bivariate exponential generating function for $A_{n+k,k}$ is given by*

$$A(x,y)=\sum_{n,k\geq 0}A_{n+k,k}\frac{x^n}{n!}\frac{y^k}{k!}=\exp(e^{x+y}-x-1).$$

Proof. Define

$$A_k(x)=\sum_{n\geq 0}A_{n+k,k}\frac{x^n}{n!}.$$

Since $A_{n,0}=V_n$ and $A_{n+1,1}=B_n$, it is clear that $A_0(x)=V(x)$ and $A_1(x)=B(x)$. From (2.2), one can derive that

$$A_k(x)=A_{k-1}(x)+A'_{k-1}(x).$$

Let $\mathcal{D}$ denote the derivative with respect to x, we have

$$A_k(x)=(1+\mathcal{D})A_{k-1}(x)=(1+\mathcal{D})^kA_0(x).$$

Then

$$\begin{aligned}A(x,y)&=\sum_{k\geq 0}A_k(x)\frac{y^k}{k!}=\sum_{k\geq 0}\frac{y^k(1+\mathcal{D})^k}{k!}A_0(x)\\&=e^{y+y\mathcal{D}}A_0(x)=e^ye^{y\mathcal{D}}A_0(x)=e^yA_0(x+y)\quad\text{(by Taylor's Theorem [16])}\\&=\exp(e^{x+y}-x-1).\end{aligned}$$

This completes the proof. □

The general formula for the Bell polynomial $B_k(x)=\sum_{j=0}^{k}S(k,j)x^j$ states that

$$B_k(x)=e^{-x}\sum_{m\geq 0}\frac{m^kx^m}{m!},$$

which, when $x=1$, produces the Dobinski formula [19] for Bell numbers

$$B_k=\frac{1}{e}\sum_{m\geq 0}\frac{m^k}{m!}.$$

Analogously, we can derive a Dobinski-type formula for $A_{n+k,k}$.

Corollary 2.2. *For any integers $n,k\geq 0$, there holds*

$$A_{n+k,k}=\frac{1}{e}\sum_{m=0}^{\infty}\frac{m^k(m-1)^n}{m!}.\tag{2.3}$$

Proof. By Theorem 2.1, one has

$$\begin{aligned}A(x,y) &= \exp(e^{x+y}-x-1)\\ &= e^{-x-1}\sum_{m\geq 0}\frac{e^{(x+y)m}}{m!}\\ &= e^{-1}\sum_{m\geq 0}\frac{1}{m!}\sum_{n\geq 0}\frac{(m-1)^n x^n}{n!}\sum_{k\geq 0}\frac{m^k y^k}{k!}\\ &= e^{-1}\sum_{n,k\geq 0}\frac{x^n}{n!}\frac{y^k}{k!}\sum_{m\geq 0}\frac{m^k(m-1)^n}{m!},\end{aligned}$$

which leads to (2.3) by comparing the coefficients of $\frac{x^n}{n!}\frac{y^k}{k!}$. □

Remark 2.3. According to the Dobinski-type formula for $A_{n+k,k}$, one can deduce the column generating function $A_k(x) = V(x)B_k(e^x)$. By extracting the coefficient of $\frac{y^k}{k!}$ from $A(x,y)$, one can also find that $A_k(x) = e^{-x}\sum_{n\geq 0} B_{n+k}\frac{x^n}{n!} = e^{-x}\mathscr{D}^k B(x)$. Then one has the relation for Bell polynomials $\mathscr{D}^k B(x) = B(x)B_k(e^x)$.

Theorem 2.4. *For any integers $n, m, k \geq 0$, there hold*

$$A_{n+m,m} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j}, \tag{2.4}$$

$$A_{n+m+k,m+k} = \sum_{j=0}^{m}\binom{m}{j}A_{n+k+j,k}. \tag{2.5}$$

Proof. Note that $A(x,y) = B(x+y)e^{-x}$ and $\frac{\partial^k}{\partial y^k}A(x,y) = A_k(x+y)e^y$ from Theorem 2.1. By equating the coefficients of $\frac{x^n y^m}{n!m!}$ in the resulting series, one can easily deduce (2.4) and (2.5). Here we provide a combinatorial proof.

(1) Let $\mathbb{S}$ denote the set of partitions of $[n+m+1]$ containing at least the singleton $m+1$, Clearly, $|\mathbb{S}| = B_{m+n}$. Let $\mathbb{S}_i$ be the subset of $\mathbb{S}$ containing another additional singleton $m+i+1$ for $1 \leq i \leq n$. Then $\overline{\mathbb{S}}_i := \mathbb{S} - \mathbb{S}_i$ is the set of partitions of $[n+m+1]$ such that $m+1$ must be a singleton and $m+i+1$ must not be a singleton, so $\bigcap_{i=1}^{n}\overline{\mathbb{S}}_i$ is just the set of partitions of $[n+m+1]$ with the largest singleton $m+1$ and hence it is counted by $A_{n+m,m}$. For any nonempty $(n-j)$-subset $\mathbb{A} \in [n]$, $\bigcap_{i\in\mathbb{A}}\mathbb{S}_i$ is the set of partitions of $[n+m+1]$ such that $m+1$ and $m+i+1$ must be singletons for all $i \in \mathbb{A}$, so it is counted by B_{m+j}. By the Inclusion–Exclusion principle, we have

$$\begin{aligned}\left|\bigcap_{i=1}^{n}\overline{\mathbb{S}}_i\right| &= \left|\mathbb{S} - \bigcup_{i=1}^{n}\mathbb{S}_i\right|\\ &= |\mathbb{S}| + \sum_{j=0}^{n-1}(-1)^{n-j}\binom{n}{j}\left|\bigcap_{i\in\mathbb{A},\,|\mathbb{A}|=n-j}\mathbb{S}_i\right|\\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j},\end{aligned}$$

which proves (2.4).

(2) A partition π of $[n+m+k+1]$ with the largest singleton $m+k+1$ can be obtained as follows. Suppose that π has exactly $m-j$ singletons in $\{k+1,\ldots,k+m\}$, there are $\binom{m}{j}$ ways to do this, so the

remainder j elements in $\{k+1,\ldots,k+m\}$ cannot be singletons in π. These j elements can be regarded as the roles that are greater than $m+k+1$ and the largest singleton $m+k+1$ plays the role as $k+1$, so under this condition there are $A_{n+k+j,k}$ ways to produce a partition π' of the remainder $n+k+j+1$ elements with the largest singleton $m+k+1$, then π' together with the $m-j$ singletons forms the desired partition π. Thus there are $\binom{m}{j}A_{n+k+j,k}$ of such partitions. Summing up all the possible cases yields (2.5). $\square$

The cases $k=0$ and $k=1$ in (2.5) produce the following corollary.

Corollary 2.5. *For any integers $n,m\geq 0$, there hold*

$$A_{n+m,m}=\sum_{j=0}^{m}\binom{m}{j}V_{n+j},$$

$$A_{n+m+1,m+1}=\sum_{j=0}^{m}\binom{m}{j}B_{n+j}. \tag{2.6}$$

Remark 2.6. The case $m:=m+1$ in (2.4), together with (2.6), produces another identity for Bell numbers

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{m+j+1}=\sum_{j=0}^{m}\binom{m}{j}B_{n+j}.$$

Spivey [21] found a generalized recurrence for Bell numbers

$$B_{n+k}=\sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}B_rS(k,j)j^{n-r},$$

and gave it a simple combinatorial proof. This recurrence has been generalized by Belbachir and Mihoubi [2], and Gould and Quaintance [9]. We also have a similar formula for $A_{n+k,k}$.

Theorem 2.7. *For any integers $n,k\geq 0$, there hold*

$$A_{n+k,k}=\sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}V_rS(k,j)j^{n-r}, \tag{2.7}$$

$$A_{n+k,k}=\sum_{r=0}^{n}\sum_{j=0}^{k}\binom{n}{r}B_rS(k,j)(j-1)^{n-r}. \tag{2.8}$$

Proof. Note that $A_k(x)=V(x)B_k(e^x)$ and $A_k(x)=B(x)B_k(e^x)e^{-x}$ from Remark 2.3. By equating the coefficients of $\frac{x^n}{n!}$ in the resulting series, one can easily deduce (2.7) and (2.8). Here we provide a combinatorial proof only for (2.7), and (2.8) can be established in a similar manner.

For the set $[n+k+1]$, one can count the number of ways to partition these $n+k+1$ elements in the following manner. Partition the set $[k]$ into exactly j blocks; there are $S(k,j)$ ways to do this. Choose an r-subset from the set $\{k+2,\ldots,n+k+1\}$ to be partitioned into new blocks, and distribute the remaining $n-r$ elements among the j blocks formed from the set $[k]$. There are $\binom{n}{r}$ ways to choose the r elements, V_r ways to partition them into new blocks without singletons, and j^{n-r} ways to distribute the remaining $n-r$ elements among the j blocks. Thus, there are $\binom{n}{r}V_rS(k,j)j^{n-r}$ of such partitions. Note that $k+1$ is always a singleton, summing over all possible values of j and r produces all ways to partition the set $[n+k+1]$ with the largest singleton $k+1$. We are done. $\square$

3. Identities involving $A_{n,k}$ and Bell numbers B_n

Theorem 3.1. *For any integer $n \geq 0$ and any indeterminate y, there holds*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k}(y+1)^k = \sum_{k=0}^{n}\binom{n}{k}y^kB_k, \tag{3.1}$$

or equivalently

$$\sum_{k=0}^{n}\binom{n}{k}A_{n,k}y^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(y+1)^kB_k. \tag{3.2}$$

Proof. Note that $A(-x, x(y+1)) = B(xy)e^x$ and $A(x, xy) = B(x(y+1))e^{-x}$ from Theorem 2.1. By equating the coefficients of $\frac{x^n}{n!}$ in the resulting series, one can easily deduce (3.1) and (3.2). Also, (3.2) can be obtained from (3.1) by setting $y := -y-1$. One can be asked to give a combinatorial proof for these two identities. $\square$

Corollary 3.2. *For any integer $n \geq 0$, there hold*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k} = 1, \tag{3.3}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}2^kA_{n,k} = B_{n+1}, \tag{3.4}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}A_{n,k}B_{k+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}B_kB_k(y), \tag{3.5}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B_kB_{k+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}A_{n,k}B_k(y), \tag{3.6}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y+k}{k}A_{n,k} = \sum_{k=0}^{n}\binom{n}{k}\binom{y}{k}B_k, \tag{3.7}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y+k}{k}B_k = \sum_{k=0}^{n}\binom{n}{k}\binom{y}{k}A_{n,k}. \tag{3.8}$$

Proof. The case $y = 0$ in (3.1) yields (3.3). The case $y = 1$ in (3.1), together with (1.2), yields (3.4).

For (3.5) and (3.6), define a linear (invertible) transformation

$$L_1(y^k) = B_k(y), \quad (k = 0, 1, 2, \ldots).$$

It is well known that $B_k(y)$ [1] satisfies the relation

$$B_{n+1}(y) = y\sum_{k=0}^{n}\binom{n}{k}B_k(y).$$

Then, we have

$$yL_1((y+1)^n) = y\sum_{k=0}^{n}\binom{n}{k}L_1(y^k) = y\sum_{k=0}^{n}\binom{n}{k}B_k(y) = B_{n+1}(y).$$

Hence (3.5) and (3.6) follow by acting yL_1 on the two sides of (3.1) and (3.2) respectively.

Similarly, if define another linear transformation

$$L_2(y^k) = \binom{y}{k}, \quad (k = 0, 1, 2, \ldots),$$

by Vandermonde's convolution identity

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n},$$

we have

$$L_2((y+1)^n) = \sum_{k=0}^{n} \binom{n}{k} L_2(y^k) = \binom{y+n}{n}.$$

Then acting L_2 on the two sides of (3.1) and (3.2) leads, respectively, to (3.7) and (3.8), another equivalent form of Theorem 3.1. □

With the Bell umbra $\mathbf{B}$ [8,17,18], given by $\mathbf{B}^n = B_n$, (1.2) may be written as $\mathbf{B}^{n+1} = (\mathbf{B}+1)^n$. By (2.4), $A_{n,k}$ can be written umbrally as

$$A_{n,k} = \mathbf{B}^k(\mathbf{B}-1)^{n-k}.$$

Setting $y := \frac{y}{1-y}$ in (3.1) and (3.2), and multiplying both sides by $y^m(1-y)^n$, we have

$$\sum_{k=0}^{n} \binom{n}{k} A_{n,k} y^m (y-1)^{(n+m-k)-m} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^{m+k} (y-1)^{(n+m)-(m+k)} B_k,$$

$$\sum_{k=0}^{n} \binom{n}{k} y^m (y-1)^{(n+m-k)-m} B_k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} y^{m+k} (y-1)^{(n+m)-(m+k)} A_{n,k},$$

which, when $y = \mathbf{B}$, produces another two identities.

Corollary 3.3. *For any integers $n, m \geq 0$, there hold*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_{n+m,m+k} B_k = \sum_{k=0}^{n} \binom{n}{k} A_{n+m-k,m} A_{n,k},$$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_{n+m,m+k} A_{n,k} = \sum_{k=0}^{n} \binom{n}{k} A_{n+m-k,m} B_k.$$

Theorem 3.4. *For any integers $n, k \geq 0$ and any indeterminate y, there holds*

$$\sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} (y+1)^{n-j} = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} y^{n-j}. \tag{3.9}$$

Proof. Note that $A(x,t)e^{x(y+1)} = B(x+t)e^{xy}$ from Theorem 2.1. By equating the coefficients of $\frac{x^n t^k}{n!k!}$ in the resulting series, one can easily deduce (3.9). Here we provide a combinatorial proof for (3.9).

Let $\mathbb{X}_{n,k} = \bigcup_{j=0}^{n} \mathbb{X}_{n,k,j}$ and $\mathbb{X}_{n,k,j}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-j)$-subset of $[k+2, n+k+1] = \{k+2, \ldots, n+k+1\}$, and each element of $\mathbb{S}$ has weight 1 or y; in other words, each element of $\mathbb{S}$ has weight $1+y$;
- π is a partition of the set $[n+k+1] - \mathbb{S}$ with the largest singleton $k+1$, and each element of $[n+k+1] - \mathbb{S}$ has weight 1.

Let $\mathbb{Y}_{n,k} = \bigcup_{j=0}^{n} \mathbb{Y}_{n,k,j}$ and $\mathbb{Y}_{n,k,j}$ denote the set of pairs $(\pi, \mathbb{S})$ such that

- $\mathbb{S}$ is an $(n-j)$-subset of $[k+2, n+k+1]$ and each element of $\mathbb{S}$ has weight y;
- π is a partition of the set $[n+k+1] - \mathbb{S}$ such that $k+1$ must be a singleton, and each element of $[n+k+1] - \mathbb{S}$ has weight 1.

The weight of $(\pi, \mathbb{S})$ is defined to be the product of the weight of each element of $[n+k+1]$. Clearly, the weights of $\mathbb{X}_{n,k}$ and $\mathbb{Y}_{n,k}$ are counted respectively by the left and right sides of (3.9).

Given any pair $(\pi, \mathbb{S}) \in \mathbb{X}_{n,k}$, $\mathbb{S}$ can be partitioned into two parts, $\mathbb{S}_1$ and $\mathbb{S}_2$ such that each element of $\mathbb{S}_1$ has weight y and each element of $\mathbb{S}_2$ has weight 1. Consider each element of $\mathbb{S}_2$ as a singleton, together with π, we obtain a partition π_1 of $[n+k+1] - \mathbb{S}_1$ such that $k+1$ is a singleton. Then the pair $(\pi_1, \mathbb{S}_1)$ lies in $\mathbb{Y}_{n,k}$.

Conversely, for any pair $(\pi_1, \mathbb{S}_1) \in \mathbb{Y}_{n,k}$, let $\mathbb{S}$ denote the union of $\mathbb{S}_1$ and the singletons of π_1 greater than $k+1$, then π_1 can be partitioned into two parts π and π' such that π is a partition of $[n+k+1]-\mathbb{S}$ with the largest singleton $k+1$ and π' is the singletons of π_1 greater than $k+1$. Then the pair $(\pi, \mathbb{S})$ lies in $\mathbb{X}_{n,k}$.

Clearly, we find a bijection between $\mathbb{X}_{n,k}$ and $\mathbb{Y}_{n,k}$, which proves (3.9). $\square$

Setting $y = 0$ and $y = 1$ in (3.9), by (2.5) in the case $k = 1$, we have the following corollary.

Corollary 3.5. *For any integers $n, k \geq 0$, there hold*

$$B_{n+k} = \sum_{j=0}^{n} \binom{n}{j} A_{k+j,k},$$

$$A_{n+k+1,n+1} = \sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} 2^{n-j}.$$

By acting yL_1 and L_2 respectively on the two sides of (3.9), we have the following corollary.

Corollary 3.6. *For any integers $n, k \geq 0$ and any indeterminate y, there hold*

$$\sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} B_{n-j+1}(y) = y \sum_{j=0}^{n} \binom{n}{j} B_{k+j} B_{n-j}(y),$$

$$\sum_{j=0}^{n} \binom{n}{j} \binom{y+n-j}{n-j} A_{k+j,k} = \sum_{j=0}^{n} \binom{n}{j} \binom{y}{n-j} B_{k+j}.$$

Corollary 3.7. *For any integers $n, k, m, i \geq 0$, there hold*

$$\sum_{j=0}^{n} \binom{n}{j} A_{k+j,k} (n-j)! = \sum_{j=0}^{n} \binom{n}{j} B_{k+j} D_{n-j}, \tag{3.10}$$

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} A_{k+j,k} A_{m+i+j,m} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} A_{n+m+i,n+m-j} B_{k+j}, \tag{3.11}$$

where D_n is the number of permutations of $[n]$ without fixed points.

Proof. The exponential generating function for D_n [22] is

$$\sum_{n\geq 0} D_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x},$$

from which, one can get

$$n! = \sum_{j=0}^{n} \binom{n}{j} D_{n-j}.$$

Let $\mathbf{D}$ be the umbra, given by $\mathbf{D}^n = D_n$, we have $n! = (\mathbf{D}+1)^n$. Then (3.10) can be obtained by setting $y = \mathbf{D}$ in (3.9).

Setting $y := \frac{y}{1-y}$ in (3.9) and multiplying both sides by $y^m(y-1)^{n+i}$, we have

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}A_{k+j,k}y^m(y-1)^{i+j} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}B_{k+j}y^{n+m-j}(y-1)^{i+j},$$

which, when $y = \mathbf{B}$, yields (3.11). $\square$

Gould and Quaintance [9] presented the identity

$$\sum_{j=0}^{m} s(m,j)B_{k+j} = \sum_{i=0}^{k}\binom{k}{i}m^{k-i}B_i,$$

which is a special case ($n = 0$) of the following three identities.

Theorem 3.8. *For any integers $n, m, k \geq 0$, there hold*

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} = \sum_{i=0}^{k}\sum_{r=0}^{n}\binom{k}{i}\binom{n}{r}m^{n+k-i-r}A_{r+i,i}, \tag{3.12}$$

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} = \sum_{i=0}^{k}\sum_{r=0}^{n}\binom{k}{i}\binom{n}{r}m^{k-i}(m-1)^{n-r}B_{r+i}, \tag{3.13}$$

$$\sum_{j=0}^{m} s(m,j)A_{n+k+j+1,n+1} = \sum_{i=0}^{k}\sum_{r=0}^{n}\binom{k}{i}\binom{n}{r}m^{k-i}(m+1)^{n-r}B_{r+i}, \tag{3.14}$$

where $s(m,j)$ is the Stirling numbers of the first kind.

Proof. We know the Bell umbra $\mathbf{B}$ satisfies $\mathbf{B}^{m+1} = (\mathbf{B}+1)^m$. Then by linearity, for any polynomial $f(x)$, we have

$$\mathbf{B}f(\mathbf{B}) = f(\mathbf{B}+1),$$

which, by induction on integer $m \geq 0$, leads to

$$\mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-m+1)f(\mathbf{B}) = f(\mathbf{B}+m). \tag{3.15}$$

By the definition of the Stirling numbers of the first kind $s(k,j)$ [5],

$$x(x-1)\cdots(x-m+1) = \sum_{j=0}^{m} s(m,j)x^j,$$

and using the umbral representation for $A_{n,k}$, we have

$$\begin{aligned}
\sum_{j=0}^{m} s(m,j)A_{n+k+j,k+j} &= \sum_{j=0}^{m} s(m,j)\mathbf{B}^{k+j}(\mathbf{B}-1)^n \\
&= \mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-m+1)\mathbf{B}^k(\mathbf{B}-1)^n \\
&= (\mathbf{B}+m)^k(\mathbf{B}-1+m)^n \\
&= \sum_{i=0}^{k}\sum_{r=0}^{n}\binom{k}{i}\binom{n}{r}m^{n+k-i-r}\mathbf{B}^i(\mathbf{B}-1)^r \\
&= \sum_{i=0}^{k}\sum_{r=0}^{n}\binom{k}{i}\binom{n}{r}m^{n+k-i-r}A_{r+i,i},
\end{aligned}$$

which proves (3.12). Similarly, one can deduce (3.13) and (3.14). $\square$

Theorem 3.9. *For any integer $n \geq 0$, there hold*

$$\sum_{k=0}^{n} A_{n,k} = B_{n+1} - V_{n+1}, \tag{3.16}$$

$$\sum_{k=0}^{n} (k+1)A_{n,k} = (n+2)B_{n+1} - V_{n+3}, \tag{3.17}$$

$$\sum_{k=0}^{n} (n-k+1)A_{n,k} = V_{n+3} - (n+2)V_{n+1}. \tag{3.18}$$

Proof. Define

$$\alpha_n(x) = \sum_{k=0}^{n} A_{n,k}x^k.$$

By (2.3), we have

$$\begin{aligned}\sum_{k=0}^{n} A_{n,k}x^k &= \frac{1}{e}\sum_{k=0}^{n} x^k \sum_{m=0}^{\infty} \frac{m^k(m-1)^{n-k}}{m!} \\ &= \frac{1}{e}\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{n} (mx)^k(m-1)^{n-k} \\ &= \frac{1}{e}\sum_{m=0}^{\infty} \frac{1}{m!} \frac{(mx)^{n+1} - (m-1)^{n+1}}{mx - m + 1},\end{aligned}$$

which, when $x = 1$, generates (3.16).

Differentiating $x\alpha_n(x)$ and then setting $x = 1$ gives

$$\begin{aligned}\sum_{k=0}^{n} (k+1)A_{n,k} &= \frac{1}{e}\sum_{m=0}^{\infty} \frac{(n+1)m^{n+1} - m^{n+1}(m-1) + (m-1)^{n+2}}{m!} \\ &= (n+1)B_{n+1} - A_{n+2,n+1} + A_{n+2,0} \\ &= (n+1)B_{n+1} - (A_{n+2,n+2} - A_{n+1,n+1}) + (B_{n+2} - A_{n+3,0}) \\ &= (n+1)B_{n+1} - (B_{n+2} - B_{n+1}) + (B_{n+2} - V_{n+3}) \\ &= (n+2)B_{n+1} - V_{n+3},\end{aligned}$$

which proves (3.17). It is routine to get (3.18) from (3.16) and (3.17). □

Remark 3.10. Canfield [4] has shown that the average number of singletons in a partition of $[n]$ is an increasing function of n. We conjecture that the average number of the largest or smallest singletons in a partition of $[n+1]$ is also an increasing function of n. That is to say, both

$$\frac{(n+2)B_{n+1} - V_{n+3}}{B_{n+1}} \quad \text{and} \quad \frac{V_{n+3} - (n+2)V_{n+1}}{B_{n+1}}$$

are increasing functions of n. One can be asked for asymptotic formulas for the above two expressions.

4. Congruence properties of $A_{n,k}$ and Bell numbers B_n

In this section, based on umbral calculus, we study the congruence properties of $A_{n,k}$ and Bell numbers B_n. Throughout this section, p refers to a prime, and unless stated otherwise, all congruences are modulo p.

Theorem 4.1. *For any integers $n, m, k \geq 0$, there holds*

$$A_{n+pm+k,k} \equiv A_{n+m+k,m+k} \pmod p.$$

Proof. Setting $x = \mathbf{B}$ in the Lagrange congruence [5],

$$x(x-1)\cdots(x-p+1) \equiv x^p - x,$$

by (3.15), for any polynomial $f(x)$, we have

$$(\mathbf{B}^p - \mathbf{B})f(\mathbf{B}) \equiv \mathbf{B}(\mathbf{B}-1)\cdots(\mathbf{B}-p+1)f(\mathbf{B}) = f(\mathbf{B}+p) \equiv f(\mathbf{B}).$$

Using the binomial congruence [5],

$$(x-1)^p \equiv x^p - 1,$$

by induction on integer $j \geq 0$, one gets

$$(\mathbf{B}^p - \mathbf{B})^j f(\mathbf{B}) \equiv f(\mathbf{B}), \tag{4.1}$$

$$(\mathbf{B}-1)^{pj} f(\mathbf{B}) \equiv \mathbf{B}^j f(\mathbf{B}). \tag{4.2}$$

Using the umbral representation for $A_{n,k}$, we have

$$\begin{aligned} A_{n+pm+k,k} &= \mathbf{B}^k(\mathbf{B}-1)^{n+pm} \\ &= (\mathbf{B}-1)^{pm}\mathbf{B}^k(\mathbf{B}-1)^n \\ &\equiv \mathbf{B}^{m+k}(\mathbf{B}-1)^n \\ &= A_{n+m+k,m+k}, \end{aligned}$$

as claimed. □

Corollary 4.2. *For any integers $n, m \geq 0$, there hold*

$$B_{n+pm} \equiv A_{n+m+1,m+1} \pmod p, \tag{4.3}$$

$$B_{n+p} \equiv B_n + B_{n+1} \pmod p, \quad \text{(Touchard's congruence [23,24])}, \tag{4.4}$$

$$A_{(n+1)p,p} \equiv B_n + B_{n+1} \pmod p, \tag{4.5}$$

$$B_{np} \equiv B_{n+1} \pmod p, \quad \text{(Comtet's congruence [5,7])}. \tag{4.6}$$

Proof. The case $k = 1$ in Theorem 4.1 leads to (4.3), which in the case $m = 1$ yields (4.4). (4.5) follows by setting $n = 0$, $m = n$, $k = p$, and (4.6) follows by setting $n = 0$, $m = n$, $k = 1$ in Theorem 4.1. □

Theorem 4.3. *For any integers $n, m, k \geq 0$, there holds*

$$A_{n+p^m+k,k} \equiv mA_{n+k,k} + A_{n+k+1,k} \pmod p.$$

In particular, the case $k = 1$ generates the generalized Touchard's congruence

$$B_{n+p^m} \equiv mB_n + B_{n+1} \pmod p.$$

Proof. By (4.1) and (4.2), when $f(x) = 1$, one has

$$\mathbf{B}^p \equiv \mathbf{B} + 1,$$

$$(\mathbf{B}-1)^p \equiv \mathbf{B}.$$

Using the little Fermat's congruence $k^p \equiv k$ [5], where k is an integer, by induction on integer $m \geq 0$, we have

$$(\mathbf{B}-1)^{p^m} \equiv \mathbf{B} + m - 1. \tag{4.7}$$

Then

$$\begin{aligned} A_{n+p^m+k,k} &= (\mathbf{B}-1)^{p^m}\mathbf{B}^k(\mathbf{B}-1)^n \\ &\equiv (\mathbf{B}+m-1)\mathbf{B}^k(\mathbf{B}-1)^n \\ &= m\mathbf{B}^k(\mathbf{B}-1)^n + \mathbf{B}^k(\mathbf{B}-1)^{n+1} \\ &= mA_{n+k,k} + A_{n+k+1,k}, \end{aligned}$$

as desired. □

Theorem 4.4. *Let $N_p = \frac{p^p-1}{p-1}$, for any integers $n, k \geq 0$, there hold*

$$\begin{aligned} &A_{n+N_p+k,k} \equiv A_{n+k,k} \pmod{p}, \\ &A_{n+N_p+k,N_p+k} \equiv A_{n+k,k} \pmod{p}, \end{aligned}$$

namely, the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ $(\bmod\ p)$ both have the period N_p.

Proof. By (4.7) and the Lagrange congruence, one has

$$(\mathbf{B}-1)^{N_p} = \prod_{j=1}^{p}(\mathbf{B}-1)^{p^{p-j}} \equiv \prod_{j=1}^{p}(\mathbf{B}-j-1) \equiv \prod_{j=0}^{p-1}(\mathbf{B}-j) \equiv 1.$$

Then

$$A_{n+N_p+k,k} = (\mathbf{B}-1)^{N_p}\mathbf{B}^k(\mathbf{B}-1)^n \equiv \mathbf{B}^k(\mathbf{B}-1)^n = A_{n+k,k}.$$

When $m = N_p$ in Theorem 4.1, one has

$$A_{n+N_p+k,N_p+k} \equiv A_{n+pN_p+k,k} \equiv A_{n+k,k},$$

where the last modular equality follows by the periodicity of $(A_{n+k,k})_{n\geq 0}$. □

Throughout the literature, the periodicity of $B_n \pmod{p}$ has been considerably investigated by many authors. Hall [10] showed that the Bell numbers (the case $k = 1$ for $(A_{n+k,k})_{n\geq 0}$ or the case $n = 0$ for $(A_{n+k,k})_{k\geq 0}$) have the period N_p, a result rediscovered by Williams [25]. Williams also showed that the minimum period is precisely N_p for $p = 2, 3$ and 5. Radoux [15] conjectured that N_p is the minimum period of the sequence B_n for any prime p. Levine and Dalton [13] showed that the minimum period is exactly N_p for $p = 7, 11, 13$ and 17. They also investigated the period for the other primes <50. Recently, Montgomery, Nahm and Wagstaff [14] showed that the minimum period is exactly N_p for most primes p below 180. For the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$, we also have the following conjecture.

Conjecture 4.5. *For any integer $k \geq 0$ and any prime p, the sequences $(A_{n+k,k})_{n\geq 0}$ and $(A_{n+k,k})_{k\geq 0}$ both have the minimum period N_p modulo p.*

Theorem 4.6. *Let $n, m, k \geq 0$ be integers and p be a prime. Then a necessary and sufficient condition that $A_{n+m+k,k} \equiv 0 \pmod{p}$ for $m = 0, 1, \ldots, p-2$, is that $A_{n+m+k,k} \equiv A_{n+pm+k,k} \pmod{p}$ for $m = 1, 2, \ldots, p-1$.*

Proof. By Theorem 4.1 and (2.5), we have

$$A_{n+pm+k,k} \equiv A_{n+m+k,m+k} = \sum_{j=0}^{m}\binom{m}{j}A_{n+j+k,k}. \tag{4.8}$$

Therefore, if $A_{n+m+k,k} \equiv 0 \pmod{p}$ for $m = 0, 1, \ldots, p-2$, we clearly have $A_{n+pm+k,k} \equiv 0$ and hence, trivially, $A_{n+pm+k,k} \equiv A_{n+m+k,k}(\equiv 0)$. When $m = p-1$ and $A_{n+j+k,k} \equiv 0$ for $j = 0, 1, \ldots, p-2$, (4.8) reduces to $A_{n+p(p-1)+k,k} \equiv A_{n+(p-1)+k,k}$.

Conversely, if $A_{n+m+k,k} \equiv A_{n+pm+k,k} \pmod p$ for $m = 1, 2, \ldots, p-1$, (4.8) is equivalent to

$$A_{n+m+k,k} \equiv A_{n+m+k,k} + \sum_{j=0}^{m-1} \binom{m}{j} A_{n+j+k,k}, \quad (m = 1, 2, \ldots, p-1),$$

which reduces to

$$0 \equiv \sum_{j=0}^{m-1} \binom{m}{j} A_{n+k+j,k}, \quad (m = 1, 2, \ldots, p-1). \tag{4.9}$$

The system (4.9) is triangular with diagonal coefficients $\binom{m}{m-1}$. The coefficient matrix is therefore nonsingular with determinant $(p-1)! \equiv -1$ by Wilson's theorem. Thus the only solution is given by $A_{n+m+k,k} \equiv 0 \pmod p$ for $m = 0, 1, \ldots, p-2$. $\square$

Theorem 4.7. *For any integer $k \geq 0$ and any prime p, there exists an integer $M_{p,k} \geq 0$ such that*

$$A_{M_{p,k}+m+k,k} \equiv 0 \pmod p, \quad (0 \leq m \leq p-2),$$

where

$$M_{p,k} \equiv 1 - (k-1)p - \frac{p^p - p}{(p-1)^2} \pmod{N_p}.$$

In other words, the sequence $(A_{n+k,k})_{n\geq 0} \pmod p$ contains a string of $p-1$ consecutive zeroes.

Proof. By Theorem 4.1 and (4.3), we have

$$A_{(n+(k-1)p-k)p+k,k} \equiv A_{n+(k-1)p,n+(k-1)p} = B_{n+(k-1)p} \equiv A_{n+k,k},$$

which, when $n = M_{p,k} + m$, where $M_{p,k}$ is an integer to be determined, produces

$$A_{(M_{p,k}+(k-1)p-k)p+pm+k,k} \equiv A_{M_{p,k}+m+k,k}.$$

By Theorems 4.4 and 4.6, it follows that $p-1$ consecutive zeroes of $(A_{n+k,k})_{n\geq 0} \pmod p$ will occur, beginning with $A_{M_{p,k}+k,k}$, if there holds

$$A_{(M_{p,k}+(k-1)p-k)p+pm+k,k} \equiv A_{(M_{p,k}+(k-1)p-k)p+m+k,k}, \quad (m = 1, 2, \ldots, p-1).$$

It is just required that the following condition holds

$$(M_{p,k} + (k-1)p - k)p + m \equiv M_{p,k} + m \pmod{N_p},$$

or, equivalently, if there holds

$$(M_{p,k} + (k-1)p - k)p = M_{p,k} + rN_p,$$

for some integer r. Using $N_p = \frac{p^p-1}{p-1} = \frac{p^p-p}{p-1} + 1$, routine calculation yields

$$M_{p,k} = 1 - (k-1)p + \frac{r+1}{p-1} + r\frac{p^p - p}{(p-1)^2}.$$

It is easy to verify by the binomial congruence that $\frac{p^p-p}{(p-1)^2}$ is always an integer. Since $M_{p,k}$ is also an integer, so we must have $r = -1 + t(p-1)$ for some integer t, from which it follows that

$$M_{p,k} = 1 - (k-1)p - \frac{p^p - p}{(p-1)^2} + tN_p.$$

Since the $p-1$ consecutive zeroes start with $A_{M_{p,k}+k,k}$, the proof is complete. $\square$

Using the same arguments, we have analogous results for the sequences $(A_{n+k,n})_{n\geq 0}$, their proofs are left to interested readers. The critical step for Theorem 4.9 is to show the congruence

$$A_{(n-1)p-k(p^{p-1}-1)+k,(n-1)p-k(p^{p-1}-1)} \equiv A_{n+k,n}.$$

This can be done by Theorems 4.1 and 4.4, and (4.3).

Theorem 4.8. *Let $n, m, k \geq 0$ be integers and p be a prime. Then a necessary and sufficient condition that $A_{n+m+k,n+m} \equiv 0 \pmod p$ for $m = 0, 1, \ldots, p-2$, is that $A_{n+m+k,n+m} \equiv A_{n+pm+k,n+pm} \pmod p$ for $m = 1, 2, \ldots, p-1$.*

Theorem 4.9. *For any integer $k \geq 0$ and any prime p, there exists an integer $U_{p,k} \geq 0$ such that*

$$A_{U_{p,k}+m+k,U_{p,k}+m} \equiv 0 \pmod p, \quad (0 \leq m \leq p-2),$$

where

$$U_{p,k} \equiv 1 + \frac{(p^{p-1}-1)k}{p-1} - \frac{p^p - p}{(p-1)^2} \pmod{N_p}.$$

In other words, the sequence $(A_{n+k,n})_{n\geq 0} \pmod p$ contains a string of $p-1$ consecutive zeroes.

Remark 4.10. Radoux [15] showed that if the period of the residues of the Bell sequence B_n is equal to N_p for a given prime p, then there exists a number c, depending on p, such that $B_{c+m} \equiv 0 \pmod p$ for $0 \leq m \leq p-2$. He also obtained the location of such a string of consecutive zeroes. Kahale [11] and Layman [12] showed respectively by two entirely different methods that this result holds without the hypothesis that N_p is the minimum period. Their result is a special case of Theorem 4.8 for $k = 1$ or of Theorem 4.9 for $k = 0$. Our proof methods are similar to Layman's.

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